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Crack Tip Fields in Functionally Graded Materials with Temperature-Dependent Properties

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Introduction

MATERIAL properties, such as modulus of elasticity and thermal conductivity, vary with temperature. They are usually regarded as constants in the thermal stress analyses of engineering materials and structures, which is approximately correct when the temperature variation is not significant. Structural components used in space vehicles, turbine engines, and reactor vessels are exposed to high temperature changes. In the thermal stress and fracture analyses of these structures, neglecting the temperature dependence of material properties may result in significant errors. Functionally graded materials (FGMs) are promising candidates for future high temperature applications. The microstructure of an FGM is gradually varied in a predetermined way thereby resulting in continuously graded macroscopic properties. Under elevated temperatures and severe temperature gradients, the properties of an FGM may vary significantly with temperature. Thus, the temperature dependence of material properties should be considered in the thermal stress and fracture analyses of FGMs in order to properly evaluate their structural integrity. Noda [1] proposed a general perturbation method to study thermal stresses in FGMs with temperature-dependent properties. Tanigawa et al. [2] investigated transient thermal stresses in a nonhomogeneous plate with temperature-dependent properties.

The characteristics of the asymptotic heat flux, stress, and deformation fields near a crack tip are closely related to crack growth criteria for predicting fracture. The present work investigates the crack tip heat flux and stress fields in FGMs with temperature-dependent properties. We first summarize basic equations for thermoelastic deformations of FGMs with temperature-dependent properties and then discuss the heat flux and stress fields and their dominance conditions, respectively.

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Basic Equations

The basic equations governing the temperature T and the Airy stress function F for thermoelastic FGMs with temperature-dependent properties undergoing plane strain deformations are

$$\nabla^2 T + \frac{1}{k} k_{,i} T_{,i} = \frac{1}{\kappa}\frac{\partial T}{\partial t} \tag{1}$$

$$\frac{1-\nu^2}{E}\nabla^2\nabla^2 F + 2\left(\frac{1-\nu^2}{E}\right)_{,i}(\nabla^2 F)_{,i} + \left(\frac{1-\nu^2}{E}\right)_{,ii}\nabla^2 F - \left(\frac{1+\nu}{E}\right)_{,ij}(\delta_{ij}\nabla^2 F - F_{,ij}) + \left[(1+\nu)\int_{T_0}^{T}\alpha(x_i, T)\mathrm{d}T\right]_{,ii} = 0 \tag{2}$$

where E is Young's modulus, ν Poisson's ratio, α the coefficient of thermal expansion, k the thermal conductivity, κ the thermal diffusivity, T_0 a reference temperature, t time, and ∇^2 the two dimensional Laplace operator. All material properties are explicit functions of spatial coordinate x_i and the temperature T, that is,

$$E = E(x_i, T), \qquad \nu = \nu(x_i, T) \qquad \alpha = \alpha(x_i, T)$$
$$k = k(x_i, T), \qquad \kappa = \kappa(x_i, T)$$

The derivatives of material properties in Eqs. (1) and (2) are

$$\begin{aligned}
\left(\frac{1-\nu^2}{E}\right)_{,i} &= \frac{\partial}{\partial x_i}\left(\frac{1-\nu^2}{E}\right) + \frac{\partial}{\partial T}\left(\frac{1-\nu^2}{E}\right)\frac{\partial T}{\partial x_i}\\
\left(\frac{1+\nu}{E}\right)_{,ij} &= \frac{\partial^2}{\partial x_i \partial x_j}\left(\frac{1+\nu}{E}\right) + \frac{\partial^2 T}{\partial x_i \partial x_j}\frac{\partial}{\partial T}\left(\frac{1+\nu}{E}\right)\\
&\quad + \frac{\partial T}{\partial x_i}\frac{\partial^2}{\partial x_j \partial T}\left(\frac{1+\nu}{E}\right) + \frac{\partial T}{\partial x_j}\frac{\partial^2}{\partial x_i \partial T}\left(\frac{1+\nu}{E}\right)\\
&\quad + \frac{\partial T}{\partial x_i}\frac{\partial T}{\partial x_j}\frac{\partial^2}{\partial T^2}\left(\frac{1+\nu}{E}\right)\\
\left(\frac{1-\nu^2}{E}\right)_{,ii} &= \frac{\partial^2}{\partial x_i \partial x_i}\left(\frac{1-\nu^2}{E}\right) + \frac{\partial^2 T}{\partial x_i \partial x_i}\frac{\partial}{\partial T}\left(\frac{1-\nu^2}{E}\right)\\
&\quad + \frac{\partial T}{\partial x_i}\frac{\partial^2}{\partial x_i \partial T}\left(\frac{1-\nu^2}{E}\right) + \frac{\partial T}{\partial x_i}\frac{\partial^2}{\partial x_i \partial T}\left(\frac{1-\nu^2}{E}\right)\\
&\quad + \frac{\partial T}{\partial x_i}\frac{\partial T}{\partial x_i}\frac{\partial^2}{\partial T^2}\left(\frac{1-\nu^2}{E}\right)\\
k_{,i} &= \frac{\partial k}{\partial x_i} + \frac{\partial k}{\partial T}\frac{\partial T}{\partial x_i}
\end{aligned} \tag{3}$$

Equation (3) implies that the gradients of material properties may be *singular* at the crack tip because the heat fluxes are generally singular. This is different from FGMs with temperature-independent properties wherein the gradients of material properties are finite. Here a material with temperature-independent properties means that the temperature dependence is neglected.

Crack Tip Heat Flux Field and Its Dominance

Consider a crack in an FGM with temperature-dependent properties. Let (r, θ) denote the polar coordinates centered at the crack tip with the crack surfaces at $\theta = \pm\pi$. By using the asymptotic expansion method, it can be shown that the governing equation of temperature field, Eq. (1), reduces to the harmonic equation because the second term on the left-hand side of the equation has a weaker singularity than the first term. Hence, the temperature and the heat flux fields have the same forms as those for homogeneous materials [3], that is,

$$T(r,\theta,t) = T_{\text{tip}}(t) - \frac{K_Q(t)}{k_{\text{tip}}}\sqrt{\frac{2r}{\pi}}\sin\frac{\theta}{2} \tag{4}$$

$$(q_r, q_\theta) = -k\left(\frac{\partial T}{\partial r}, \frac{1}{r}\frac{\partial T}{\partial \theta}\right) = \frac{K_Q(t)}{\sqrt{2\pi r}}\left(\sin\frac{\theta}{2}, \cos\frac{\theta}{2}\right) \tag{5}$$

where K_Q is the heat flux intensity factor and the subscript "tip" stands for quantities at the crack tip.

Asymptotic fields (4) and (5) are valid only at points very close to the crack tip as compared with the crack length and other characteristic dimensions of the cracked body. While the gradient of the thermal conductivity and its temperature dependence do not influence the square root singularity of the heat flux field, they may affect the size of the region in which Eqs. (4) and (5) hold. By using an asymptotic analysis method, we can show that the dominance condition of the crack tip fields (4) and (5) is

$$\left|\frac{\partial k}{\partial x_i}\right| + \left|\frac{\partial k}{\partial T}\frac{K_Q}{k_{\text{tip}}}\right|\frac{1}{\sqrt{2\pi r}} \ll \frac{k}{r} \tag{6}$$

When the temperature dependence of thermal conductivity is neglected, Eq. (6) reduces to

$$\left|\frac{\partial k}{\partial x_i}\right| \ll \frac{k}{r} \tag{7}$$

When the thermal conductivity is not explicitly dependent on the spatial coordinates, Eq. (6) reduces to the dominance conditions for materials with temperature-dependent thermal conductivity [4]

$$\left|\frac{\partial k}{\partial T}\frac{K_Q}{k_{\text{tip}}}\right|\frac{1}{\sqrt{2\pi r}} \ll \frac{k}{r} \tag{8}$$

Crack Tip Stress Field and the *K* Dominance

This section considers the crack tip stress field and its dominance for thermoelastic FGMs with temperature-dependent properties. Again, using the asymptotic expansion method, and considering the gradients of material properties in Eq. (3) and the heat flux field, Eq. (5), we can show that the square root singularity of the classical fracture mechanics still prevails and the crack tip stress field has the standard form

$$\sigma_{ij}(r,\theta,t) = \frac{1}{\sqrt{2\pi r}}\left\{K_{\text{I}}(t)\tilde{\sigma}_{ij}^{(1)}(\theta) + K_{\text{II}}(t)\tilde{\sigma}_{ij}^{(2)}(\theta)\right\} \tag{9}$$

where K_{I} and K_{II} are the mode-I and the mode-II stress intensity factors, respectively, and $\tilde{\sigma}_{ij}^{(1)}(\theta)$ and $\tilde{\sigma}_{ij}^{(2)}(\theta)$ are the corresponding angular variations of the stress field. Although the form of the crack tip stress field is not influenced by the gradient of the elastic modulus and its temperature dependence, the size of the region in which Eq. (9) dominates will be affected. Jin and Batra [5] have given the following K-dominance conditions for FGMs without consideration of the temperature dependence of material properties

$$\left|\frac{\partial E}{\partial x_i}\right| \ll \frac{E}{r}, \qquad \left|\frac{\partial^2 E}{\partial x_i \partial x_j}\right| \ll \frac{E}{r^2} \tag{10}$$

For FGMs with temperature-dependent properties, inequalities (10) still hold if the partial differentiations in them are replaced by differentiations considering the effect of temperature T as implied by Eq. (3). For FGMs with temperature-dependent properties, the gradients of Young's modulus are given by

$$E_{,i} = \frac{\partial E}{\partial x_i} + \frac{\partial E}{\partial T}\frac{\partial T}{\partial x_i}$$

$$E_{,ij} = \frac{\partial^2 E}{\partial x_i \partial x_j} + \frac{\partial^2 T}{\partial x_i \partial x_j}\frac{\partial E}{\partial T} + \frac{\partial T}{\partial x_i}\frac{\partial^2 E}{\partial x_j \partial T} + \frac{\partial T}{\partial x_j}\frac{\partial^2 E}{\partial x_i \partial T} + \frac{\partial T}{\partial x_i}\frac{\partial T}{\partial x_j}\frac{\partial^2 E}{\partial T^2} \tag{11}$$

The K-dominance conditions for FGMs with temperature-dependent properties thus become

$$\left|\frac{\partial E}{\partial x_i}\right| + \left|\frac{\partial E}{\partial T}\frac{K_Q}{k_{\text{tip}}}\right|\frac{1}{\sqrt{2\pi r}} \ll \frac{E}{r} \tag{12}$$

$$\left|\frac{\partial^2 E}{\partial x_i \partial x_j}\right| + \left|\frac{\partial^2 E}{\partial x_i \partial T}\frac{K_Q}{k_{\text{tip}}}\right|\frac{1}{\sqrt{2\pi r}} + \left|\frac{\partial^2 E}{\partial T^2}\right|\left(\frac{K_Q}{k_{\text{tip}}}\right)^2\frac{1}{2\pi r} \ll \frac{E}{r^2} \tag{13}$$

Conclusions

We have shown that the square root singular heat flux and stress fields still prevail in the crack tip region in thermoelastic FGMs with temperature-dependent properties. The size of the region in which the stress intensity factors dominate, however, will be influenced by material nonhomogeneities and the temperature dependence of material properties. An approximate theoretical estimate of the effect is given in the present work. The K-dominance zone will become smaller for FGMs with steeper gradients of properties in the crack tip region and stronger heat flux intensities. The latter effect results from the temperature dependence of material properties and the singularity of the heat flux field at the crack tip.

It is important for the K-dominance conditions (6), (12), and (13) to be checked in the numerical modeling of crack tip fields. A finer mesh and appropriate schemes for material property computation may be required to accurately evaluate the second order derivatives of thermal conductivity and Young's modulus.

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